Multifractality at the quantum Hall transition: Beyond the parabolic paradigm

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We present an ultra-high-precision numerical study of the spectrum of multifractal exponents Δ_q characterizing anomalous scaling of wave function moments $\langle|\psi|^{2q}\rangle$ at the quantum Hall transition. The result reads $\Delta_q = 2q(1-q)[b_0 + b_1(q-1/2)^2 + \ldots]$, with $b_0 = 0.1291 \pm 0.0002$ and $b_1 = 0.0029 \pm 0.0003$. The central finding is that the spectrum is not exactly parabolic, $b_1 \neq 0$. This rules out a class of theories of Wess-Zumino-Witten type proposed recently as possible conformal field theories of the quantum Hall critical point.

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Introduction The quantum Hall effect is a famous macroscopic quantum phenomenon [1, 2] whose discovery gave rise to one of the most active research areas in condensed matter physics of last three decades. The plateaus with quantized values of the Hall conductivity are separated by quantum Hall transitions, which represent a celebrated example of a quantum critical point in a disordered electronic system (for a recent review, see Ref. 3). Identification of the critical field theory of the integer quantum Hall transition remains a major unsolved problem of condensed matter physics.

One of the key characteristics of the quantum Hall transition point is the multifractality spectrum governing fluctuations of amplitudes of critical wave functions. Specifically, the moments of wavefunctions scale with system size, L, with a set of anomalous exponents Δ_q,

$$\langle|\psi(\mathbf{r})|^{2q}\rangle / \langle|\psi(\mathbf{r})|^2\rangle^q \sim L^{-\Delta_q} \ . \qquad (1)$$

(The angular brackets denote the ensemble averaging.) Equivalently, one often characterizes multifractality by a closely related set of exponents, $\tau_q \equiv d(q-1) + \Delta_q$, or by its Legendre transform $f(\alpha)$ ("singularity spectrum"), $\alpha_q = \tau'_q$, $f(\alpha_q) = q\alpha_q - \tau_q$. Here d is the system dimensionality (while $d = 2$ for the present case of the quantum Hall transition, we find it useful to keep it as d in formulas below), and the prime denotes the q-derivative.

Zirnbauer[4] and Tsvelik[5, 6] conjectured that the conformal theory of the quantum Hall critical point is of the Wess-Zumino-Witten type. These proposals imply[7] that the spectrum Δ_q is parabolic, i.e., that γ_q defined according to

$$\Delta_q = \gamma_q q(1-q) \ . \qquad (2)$$

is in fact q-independent[8], $\gamma_q = \gamma$. An accurate numerical analysis of the multifractal spectrum plays therefore a crucial role for identification of the critical theory.

A high-accuracy evaluation of the multifractality spectrum was carried out in our earlier work [9]. For this purpose, we modelled systems of a much larger size than in preceding works and performed averaging over a large ensemble of wave functions, as well as a thorough analysis of finite-size effects. It was found that the spectrum is close-to-parabolic, $\Delta_q \simeq \gamma q(1-q)$ with $\gamma = 0.262 \pm 0.003$, thus showing that, if deviations from parabolicity are present, they are rather small (of the order of 1%). While the data of Ref. 9 were showing some indications for such deviations, they were smaller than the numerical uncertainties. The latter originate from statistical noise (limited size of the data set) and from finite size effects affecting the scaling relation (1) which is used to extract Δ_q.

The goal of the present Letter is to determine the Δ_q spectrum with an ultra-high precision and to give an ultimate answer on the question "Is the multifractality spectrum of the quantum Hall transition strictly parabolic?" For this purpose, we improve upon the earlier numerical analysis in two different ways. First, we utilize a statistical ensemble that contains approximately ten times more samples than the one used before[9]. Second, we employ a recently discovered [10] "reciprocity relation",

$$\Delta_q = \Delta_{1-q}, \qquad (3)$$

for a better control of finite-size corrections.

Relation (3) implies a symmetry of the Δ_q spectrum around the point q=1/2. Consequently, an expansion of γ_q about this point has a form

$$\gamma_q/d = b_0 + b_1(q-1/2)^2 + b_2(q-1/2)^4 \ldots . \qquad (4)$$

To verify or to exclude parabolicity of Δ_q, the prefactor b_1 of the quadratic term (and possibly those of higher order terms) in Eq. (4) should be determined numerically. This is the purpose of the present work. We will provide numerical evidence that the corrections to parabolicity do not vanish. Specifically, we obtain b_0=0.1291 ± 0.0002 and b_1=0.0029 ± 0.0003, the non-zero b_1 implying that the parabolcity is not exact. The corresponding value of α_0 [position of the apex of the singularity spectrum $f(\alpha)$] is $\alpha_0 = d + \gamma_0 = 2.2596 \pm 0.0004$.

Method: In order to find the critical eigenstates, we employ the same numerical strategy that has been developed before[9]. We determine the lattice time evolution operator U for the Chalker-Coddington network

model[11, 12] with periodic boundary conditions and $N{=}2L^d$ nodes. Eight eigenstates with eigenvalues closest to unity are found with a standard sparse matrix package[13, 14, 15] from exact diagonalization of U. We study systems with $L{=}16, 32, 64, \ldots, 1024$ with $\sim 10^6$ samples for the smallest sizes and $\sim 10^4$ for the largest ones. The statistical analysis proceeds via calculating the average inverse participation ratios,

$$P_q = \left\langle \int |\psi^2|^q \right\rangle \tag{5}$$

which obey the scaling law

$$P_q = c_q(N)\; N^{-(q-1)-\Delta_q/d}. \tag{6}$$

The coefficients c_q become independent of N in the limit $N \to \infty$.

As an alternative approach, we consider the ratio

$$L_q = \frac{\langle \int |\psi^2|^q \;\ln |\psi|^2 \rangle}{\langle \int |\psi^2|^q \rangle} \equiv (\ln P_q)' = -\frac{\alpha_q}{d} \ln N + (\ln c_q)' \,, \tag{7}$$

whose scaling yields the exponent $\alpha_q = \Delta_q' + d$. In this way, the exonent α_q is studied directly, i. e. without invoking a numerical differentiation which can significantly increase the error bars. In analogy with Eq. (4), we can expand Δ_q' around $q = 1/2$,

$$\Delta_q' = (1-2q)\;\tilde{\gamma}_q, \qquad \tilde{\gamma}_q/d = a_0 + a_1(q - \frac{1}{2})^2 + \ldots. \tag{8}$$

The coefficients of both expansions are related via $a_0{=}b_0{-}b_1/4$, $a_1{=}2b_1{-}b_2/2$, ...

The averages entering Eqs. (5) and (7) are readily obtained numerically. It is beneficial to perform the scaling analysis of the ratio (7) in addition to that of Eq. (5) for several reasons. First, the curvature of Δ_q is more clearly seen in the q-derivative, Δ_q'. Second, the finite-size corrections are different in the cases of Eqs. (5) and (7), so that an agreement between the obtained exponents provides an additional confirmation of the validity of the $N \to \infty$ extrapolation procedure. Also, the relation $a_1{=}2b_1{-}b_2/2$ allows one to extract the coefficient b_2 of the quartic term in Eq. (4) out of parabolic fits for γ_q and $\tilde{\gamma}_q$.

Numerical results: We begin the analysis of our numerical results by verifying the reciprocity relation (3). To this end, we consider the ratio

$$r_q{=}N^{2q-1}\frac{P_q}{P_{1-q}} = N^{(\Delta_{1-q}-\Delta_q)/d}\frac{c_q(N)}{c_{1-q}(N)}\,. \tag{9}$$

The reciprocity relation (3) implies that the leading powers should cancel, so that r_q exhibits only subleading corrections in $1/N$. The log-linear plot, Fig. 1, upper row, shows that r_q saturates in the large N-limit with a very well defined asymptotic value. Thus, we confirm reciprocity for the exponent spectrum of the integer quantum Hall effect, as expected. [10] Since the exponent relation must hold only in the asymptotic regime, we can draw another conclusion which is important for the subsequent analysis: the observed saturation of r_q provides evidence that our numerically accessible sample sizes are indeed large enough in order to be able to study the true asymptotics.

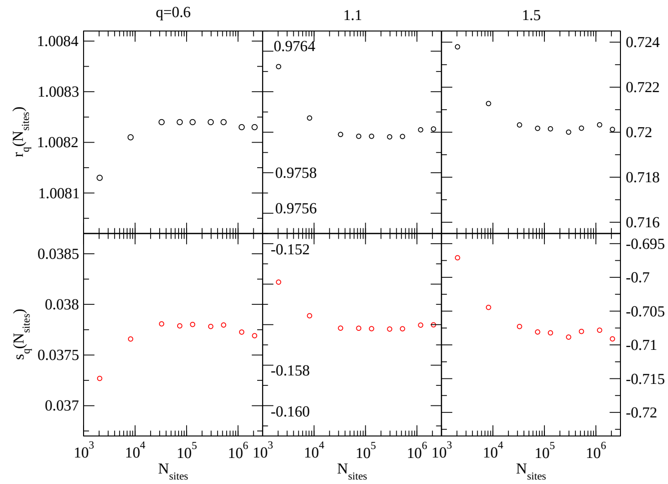

FIG. 1: Upper row: ratio r_q of inverse participation numbers P_q and P_{1-q} [Eq. (9)] at q=0.6, 1, 1.5 (from left to right). The flat asymptotics indicates validity of the reciprocity relation, $\Delta_q{=}\Delta_{1-q}$. Small deviations from the constant behavior at largest system sizes are due to residual statistical noise. Lower row: analogous plots for the logarithmic derivative $s_q = (\ln r_q)'$ defined in Eq. (10).

Similar to r_q, we consider the logarithmic derivative

$$s_q \equiv (\ln r_q)' = \frac{2d - \alpha_q - \alpha_{1-q}}{d} \ln N + (\ln c_q c_{1-q})', \tag{10}$$

which also saturates well inside the numerical window, see Fig. 1, lower row. Thus, the true asymptotics of α_q may be studied by means of Eq. (7) with available system sizes, too.

Having gone through important prerequisites, we now turn to the analysis of the main data. In order to determine a set of relatively small exponents, Δ_q, to an accuracy considerably better than 1%, we have developed the following procedure. In each panel of Fig. 2 we plot for fixed q a family of curves labelled by a parameter δ,

$$F_q(N) = P_q N^{q-1+\delta(1-q)q}. \tag{11}$$

For the particular family member, for which $F_q(N)$ becomes independent of N in the limit of large N, we can conclude that $\delta{=}\gamma_q/d$. From such a procedure we extract the function γ_q without having to resort to any (multi-parameter) fitting procedure. Similarly, by studying yet another family of curves,

$$\tilde{F}_q(N) = L_q + \ln N[1 + (1-2q)\tilde{\delta}]\,, \tag{12}$$

we have direct access also to the function $\tilde{\gamma}_q$, see Eq. (8), without the need for numerical differentiation.

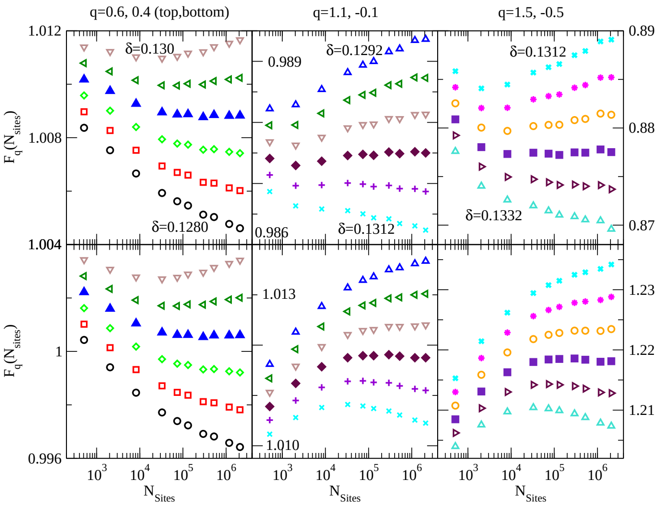

FIG. 2: Family of curves $F_q(N) = P_q N^{q-1-\delta(q-1)q}$ for q=0.6, 1.1, 1.5 (top; left,center,right) and q=0.4, −0.1, −0.5 (bottom; left,center,right). Each data set is labelled by a parameter δ, which increases from a minimum to a maxium value (given in the upper plots) in steps of 0.0004. The value of δ for which $F_q(N)$ is flat determines the anomalous exponent, Δ_q=$d\delta q(1-q)$. Such saturating data sets are marked with filled symbols (left: △; center: ◇; right: □); typical error in the corresponding value of δ does not exceed 0.001. The change in symbols (i.e. in δ) for saturating data sets illustrates a q-dependence of γ_q and thus gives direct, unprocessed evidence of nonvanishing quartic terms in Δ_q.

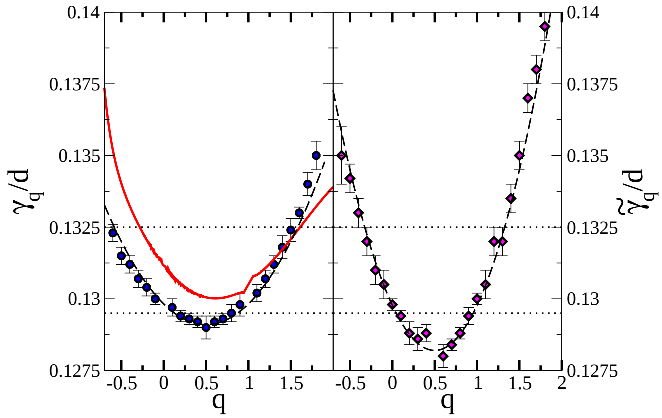

FIG. 3: The exponents γ_q=$\Delta_q/q(1-q)$ (○) and $\tilde{\gamma}_q$=$\Delta_q'/(1-2q)$ (◇) obtained from Fig. 2 and analogous analysis for other values of q. The curvature in γ_q and $\tilde{\gamma}_q$ implies that the multifractal spectrum Δ_q is not parabolic. Also shown are results of the earlier work [9] (solid line). Dotted horizontal lines indicate earlier error bars in the regime $0 \le q \le 1$. Dashed lines represent parabolic fits with b_0=0.1291 ± 0.0002, b_1=0.0029 ± 0.0003 (left) and a_0=0.1282 ± 0.0001, a_1=0.0063 ± 0.0005 (right). Combination of these data yields a rough estimate of the quartic term, $b_2 = -0.001 \pm 0.001$.

The functions γ_q and $\tilde{\gamma}_q$ representing the main result of this paper are displayed in Fig. 3. Also shown is $\Delta_q/q(1-q)$ as derived from the earlier evaluation of the exponents [9] (the size of corresponding error bars is indicated by dotted lines). Figure 3 clearly shows that the curvature in γ_q, which apparently has already left its trace in the earlier data, now fully reveals itself thanks to the reduced error bars. Even more pronounced is the resulting structure in the derivative $\tilde{\gamma}_q$. It is reassuring to notice that the new data confirm our previous finding for b_0 but provide a much better accuracy: b_0=0.1291 ± 0.0002. Our new result for the curvature of γ_q is b_1=0.0029 ± 0.0003, which is clearly non-vanishing. These results are in full agreement with those obtained by a fit to $\tilde{\gamma}_q$ (see the caption to Fig. 3). We thus conclude that, although the curvature of γ_q is numerically rather small (b_1 is approximately 50 times smaller than b_0), it is non-zero: the multifractality spectrum Δ_q of the quantum Hall transition is not parabolic.

Surface exponents: So far a network model with torus geometry (i.e. without boundary) has been considered. Recently, it has been shown [16] that wavefunction fluctuations near surfaces exhibit their own multifractal spectrum with exponents Δ_q^{s}, defined in full analogy to Eq. (1) via

$$\langle|\psi|^{2q}\rangle_{\mathrm{s}}/\langle|\psi|^2\rangle_{\mathrm{s}}^q \sim L^{-\Delta_q^{\mathrm{s}}}\,, \qquad (13)$$

with the average $\langle\ldots\rangle_{\mathrm{s}}$ performed over the boundary sites only. In general, the surface exponents Δ_q^{s} are not related to their bulk counterparts in any simple manner. We parametrize the surface spectrum in analogy with the bulk case, Eqs. (2), (4), (8), labelling the corresponding parameters by a superscript "s". The results for γ_q^s and $\tilde{\gamma}_q^s$ are shown in Fig. 4. It is seen that the non-parabolicity of the multifractality spectrum (difference of γ_q^s and $\tilde{\gamma}_q^s$ from a constant) is present at the boundary as well and is in fact considerably more pronounced that in the bulk. The ratio R_q=$\gamma_q^{\mathrm{s}}/\gamma_q$ has a clear q-dependence, with a minimum at the symmetry point $q = 1/2$, where it takes the value $R_{1/2}$=1.434 ± 0.005. It is also worth noticing that R_q is appreciably smaller than 2, a value naturally expected for critical theories expressed in terms of a free bosonic field.

Conclusions: To summarize, we have studied numerically the wave function statistics at the quantum Hall critical point. We have verified that the reciprocity relation (3) holds and have used it to control systematic errors related to the finite-size effects. In combination with a very large size of the statistical ensemble, this has allowed us to reach unprecedented accuracy in determination of the multifractality spectrum Δ_q, with the error bars reduced by almost an order of magnitude compared to the earlier work. The result shown in Fig. 3 reads $\Delta_q = 2q(1-q)[b_0 + b_1(q-1/2)^2 + b_2(q-1/2)^4 + \ldots]$, with $b_0 = 0.1291 \pm 0.0002$, $b_1 = 0.0029 \pm 0.0003$, and $b_2 = -0.001 \pm 0.001$. The obtained spectrum shows clear non-parabolicity, $b_1 \neq 0$, thus excluding a broad class of theories of the Wess-Zumino-Witten type as candidates in the conformal field theory of the quantum Hall tran-

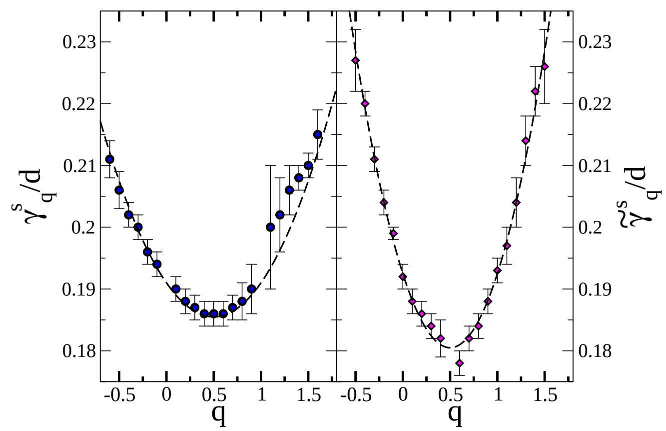

FIG. 4: The surface exponents $\gamma^s_q = \Delta^s_q/q(1-q)$. Data are presented in the form analogous to the bulk plot, Fig. 3; the curvature is even more pronounced for the surface exponents. Dashed lines indicate parabolic fits with b^s_0=0.1855 $\pm$ 0.0005, b^s_1=0.022 $\pm$ 0.002 (left) and a^s_0=0.1805 $\pm$ 0.001, a^s_1=0.048 $\pm$ 0.003 (right). The resulting estimate for b^s_2 is $b^s_2 = -0.008 \pm 0.01$.

sition. These results are corroborated by the analysis of the surface mutlifractality. While completing this work, we learnt about an independent study by Obuse *et al.* [17] who focussed on the surface multifractality spectrum and came to the same conclusions.

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